Applications of Fourier Transform Methods to Boundary Value Problems

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Abstract

In this paper, basic concepts of Fourier transforms are introduced. Then properties of Fourier transform methods are explained. Finally, applications of Fourier transform methods to heat equation and wave equation are presented with some examples.

Keywords: Fourier transforms, Convolution, Parseval's identity

Introduction

Joseph Fourier, a French mathematician, invented a method called Fourier transform in 1801, to explain the flow of heat around an anchor ring. Since then, it has become a powerful tool in diverse fields of science and engineering.

The purpose of this paper is to solve boundary value problems by using Fourier transform method. This paper is divided into four sections. Some basic concepts of Fourier transforms are firstly presented. Then, the properties of Fourier transforms are proved by using the concepts. Next convolution and Parseval identity required for the application are discussed. Fourier transform method is finally applied to heat equation and wave equation.

Fourier Transforms

If $f(x)$ is defined in (a,b), the integral transform of $f(x)$ with the Kernel $K(s,x)$ is defined by

$$F(s) = \bar{f}(s) = \int_a^b f(x)K(s,x)dx \tag{1}$$

if the integral exists.

Here, $K(s,x)$ is called the Kernel of the transform while *a*, *b* are fixed limits. If *a*, *b* are finite, the transform is finite and if *a*, *b* are infinite, it is infinite transform.

If $$K(s,x) = \begin{cases} e^{-sx}, & \text{for} \quad x \geq 0 \\ 0, & \text{for} \quad x < 0, \end{cases} \tag{2}$$

and $$F(s) = \int_a^b f(x)K(s,x)dx,$$

it may be possible to get $f(x)$ as

$$f(x) = \int_c^d F(s)H(s,x)ds\,. \tag{3}$$

Equation (3) is called the inversion formula for (1). In (1), if *F*(*s*) is known while $f(x)$ is unknown, it may be regarded as an integral equation.

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From the general integral transform definition, we can get various integral transforms by properly defining the Kernel.

Infinite Fourier transform is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

and inversion formula is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds\,.$$

Infinite Fourier cosine transform is

$$F_c(s) = \overline{f}_c(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$$

and inversion formula is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx\, ds\,.$$

Infinite Fourier sine transform is

$$F_S(s) = \overline{f}_S(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

and inversion formula is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_S(s) \sin sx\, ds\,.$$

Properties of Fourier Transforms

Theorem 1

Fourier transform is linear. i.e., $F\left[af(x) + bg(x)\right] = aF\left[f(x)\right] + bF\left[g(x)\right]$ where F stands for Fourier transform.

Proof:

$$F\left[af(x) + bg(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(af(x) + bg(x)\right) e^{isx} dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= aF\left[f(x)\right] + bF\left[g(x)\right]. \qquad \blacksquare$$

Theorem 2

If $F\left[f(x)\right] = F(s)$, then $F\left[f(x-a)\right] = e^{isa} F(s)$.

Proof:

$$F\left[f(x-a)\right] = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$$

Putting x – a = t, we get

$$F\left[f(x-a)\right] = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} f(t)e^{i(a+t)s}dt$$

$$= e^{ias}\frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} f(t)e^{its}dt$$

$$= e^{ias}F(s). \qquad \blacksquare$$

Theorem 3

If $F\left[f(x)\right]=F(s)$, then $F\left[f(ax)\right]=\frac{1}{|a|}F(\frac{s}{a})$, where $a\neq 0$.

Proof:

$$F\left[f(ax)\right] = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} e^{isx}f(ax)dx.$$

Putting $ax=t$ and $a>0$, we get

$$F\left[f(ax)\right] = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} e^{i(\frac{s}{a})t}f(t)\frac{dt}{a}$$

$$= \frac{1}{a}F(\frac{s}{a}) \text{ if } a>0.$$

$$F\left[f(ax)\right] = \frac{1}{\sqrt{2p}}\int_{\infty}^{-\infty} e^{i(\frac{s}{a})t}f(t)\frac{dt}{a} \quad \text{if } a<0$$

$$= -\frac{1}{a}F(\frac{s}{a}) \text{ if } a<0.$$

So, $$F\left[f(ax)\right] = \frac{1}{|a|}F(\frac{s}{a}). \qquad \blacksquare$$

Theorem 4

$$F\left\{e^{iax}f(x)\right\} = F(s+a).$$

Proof:

$$F\left\{e^{iax}f(x)\right\} = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} e^{i(s+a)x}f(x)dx$$

$$= F(s+a). \qquad \blacksquare$$

Theorem 5

If $F\{f(x)\} = F(s)$, then $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s-a)+F(s+a)\right]$.

Proof:

$$
\begin{aligned}
F\{f(x)\cos ax\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\frac{e^{iax}+e^{-iax}}{2}\,dx \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)(e^{iax}+e^{-iax})\,dx\right] \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\,dx\right] \\
&= \frac{1}{2}\left[F(s+a)+F(s-a)\right].
\end{aligned}
$$

■

Theorem 6

If $F\{f(x)\} = F(s)$, then $F\left[x^n f(x)\right] = (-i)^n \frac{d^n}{ds^n}F(s)$.

Proof:

We have

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx.$$

Differentiating with respect to s both sides, n times,

$$
\begin{aligned}
\frac{d^n F(s)}{ds^n} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (ix)^n e^{isx} f(x)\,dx \\
&= (i)^n \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{isx} f(x)\,dx \\
&= (i)^n F\left[x^n f(x)\right].
\end{aligned}
$$

Therefore, $$F\left[x^n f(x)\right] = (-i)^n \frac{d^n}{ds^n} F\left[(s)\right].$$ ■

Theorem 7

$F\left[f'(x)\right] = -isF(s)$ if $f(x)\to 0$ as $x\to\pm\infty$.

Proof:

$$
\begin{aligned}
F\left[f'(x)\right] &= \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} e^{isx} f'(x)dx \\
&= \frac{1}{\sqrt{2p}}\left[\left\{e^{isx} f(x)\right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)e^{isx}(is)dx\right] \\
&= \frac{1}{\sqrt{2p}}\left[\left\{e^{isx} f(x)\right\}_{-\infty}^{\infty} - is\int_{-\infty}^{\infty} f(x)e^{isx}dx\right] \\
&= -is\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx .
\end{aligned}
$$

If $f(x)\to 0$ as $x\to\pm\infty$, then

$$F\left[f'(x)\right] = -isF(s). \qquad \blacksquare$$

Theorem 8

$$F\left\{\int_a^x f(x)dx\right\} = \frac{F(s)}{(-is)}.$$

Proof:

Let $\mathrm{f}(x) = \int_a^x f(x)dx .$

Then, $\mathrm{f}'(x) = f(x).$

$$
\begin{aligned}
F\left[\phi'(x)\right] &= -isF(\mathrm{f}(x)) \\
&= -isF\int_a^x f(x)dx . \\
F\left(\int_a^x f(x)dx\right) &= \frac{1}{-is}F\left[\mathrm{f}'(x)\right] \\
&= \frac{1}{-is}F\left[f(x)\right] \\
&= \frac{F(s)}{-is}. \qquad \blacksquare
\end{aligned}
$$

Convolution and Parseval's Identity

Theorem 9

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

That is, $F\left[f(x)*g(x)\right] = F(s)\cdot G(s) = F\left[f(x)\right]\cdot F\left[g(x)\right].$

Proof:

$$
\begin{aligned}
F\left[f*g\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)e^{ixs}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)g(x-t)dt\right)e^{ixs}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x-t)e^{ixs}dx\right)dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)F\left[g(x-t)\right]dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{its}G(s)\,dt \\
&= G(s)\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{its}dt \\
&= G(s)\cdot F(s).
\end{aligned}
$$

By inversion, we get

$$
\begin{aligned}
F^{-1}\left[F(s)G(s)\right] &= f*g \\
&= F^{-1}\left[F(s)\right]*F^{-1}\left[G(s)\right]. \qquad \blacksquare
\end{aligned}
$$

Parseval's identity

If $F\left[\,s\,\right]$ is the Fourier transform of $f(x)$. then $\int_{-\infty}^{\infty}\left|f(x)\right|^2dx=\int_{-\infty}^{\infty}\left|F(s)\right|^2ds$.

Proof:

$$
\overline{F(s)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f(x)}e^{-isx}dx
$$

Putting $x=-v$, we get

$$
\begin{aligned}
\overline{F(s)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f(-v)}e^{isv}\,dv \\
&= F\left[\overline{f(-v)}\right] \\
&= F\left[\overline{f(-x)}\right] \\
F\left[f(x)*g(x)\right] &= F(s)G(s) \\
f*g &= F^{-1}\left[F(s)G(s)\right] \\
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)g(x-t)dt &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)G(s)e^{-isx}ds.
\end{aligned}
$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = \int_{-\infty}^{\infty} F(s)G(s)ds . \tag{4}$$

Since it is true for all $g(t)$, we take $g(t) = \overline{f(-t)}$.

So, $g(-t) = \overline{f(t)}$.

$$G(s) = F\left[g(t)\right] = F\left[\overline{f(-t)}\right] = \overline{F(s)}.$$

Using this in (4), we get

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)}dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}ds .$$

So, $$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} \left|F(s)^2\right| ds .$$ ■

Applications of Fourier Transform Methods to Wave Equation and Heat Equation

Example

We can solve $\dfrac{\partial^2 u}{\partial t^2} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t \geq 0$ with conditions $u(x,0) = f(x)$, $\dfrac{\partial u(x,0)}{\partial t} = g(x)$ and assuming u, $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$.

Let $\bar{u}$ be Fourier transform of *u* with respect to *x*. Taking Fourier transform on both sides of given equation, we get

$$\frac{d^2\bar{u}}{dt^2} = \alpha^2(-s^2\bar{u}),$$

$$\frac{d^2\bar{u}}{dt^2} + \alpha^2 s^2 \bar{u} = 0.$$

Auxiliary equation is

$$m^2 + \alpha^2 s^2 = 0$$

$$m = \pm i\alpha s .$$

So,

$$\bar{u}(s,t) = Ae^{i\alpha st} + Be^{-i\alpha st}. \tag{5}$$

Since $u(x,0) = f(x)$ and $\dfrac{\partial u(x,0)}{\partial t} = g(x)$,

$$\bar{u}(s,0) = F(s) \text{ and } \frac{d\bar{u}(s,0)}{dt} = G(s) .$$

So, $\bar{u}(s,0) = A + B = F(s)$ and $\dfrac{d\bar{u}(s,0)}{dt} = i\alpha s(A-B) = G(s)$.

Then, $A - B = \dfrac{G(s)}{i\alpha s}$.

Hence, $A = \dfrac{1}{2}\left[F(s) + \dfrac{G(s)}{i\alpha s}\right]$,

$$B = \frac{1}{2}\left[F(s) - \frac{G(s)}{i\alpha s}\right].$$

Using these values in (5), we get

$$\bar{u}(s,t) = \frac{1}{2}\left[F(s) + \frac{G(s)}{i\alpha s}\right]e^{i\alpha st} + \frac{1}{2}\left[F(s) - \frac{G(s)}{i\alpha s}\right]e^{-i\alpha st}.$$

Hence,

$$u(x,t) = \frac{1}{2}\left[f(x-\alpha t) - \frac{1}{\alpha}\int_{\alpha}^{x-\alpha t} g(\theta)\,d\theta\right] + \frac{1}{2}\left[f(x+\alpha t) + \frac{1}{\alpha}\int_{\alpha}^{x+\alpha t} g(\theta)\,d\theta\right].$$

Example

We can solve the diffusion equation $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$ with the conditions, $u(x,0) = f(x)$, and $\dfrac{\partial u}{\partial x}$, u tend to zero as x tend to $\pm\infty$.

Fourier transform of $u(x,t)$ is

$$\bar{u}(s,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\,e^{isx}\,dx.$$

Fourier transform of given differential equation is

$$\begin{aligned} K(-s^2\,\bar{u}(s,t)) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{\partial u}{\partial t} e^{isx}\,dx \\ &= \frac{d}{dt}\left[F(u)\right] \\ &= \frac{d\bar{u}}{dt}. \end{aligned}$$

So, $$\frac{d\bar{u}}{dt} + s^2 K\bar{u} = 0.$$

Then $$\bar{u}(s,t) = c\,e^{-s^2kt}. \tag{6}$$

Since $u(x,0) = f(x)$,

$$\bar{u}(s,0) = F(s). \tag{7}$$

Using (7) in (6), we get

$$c = F(s).$$

So, $$\bar{u}(s,t) = F(s)e^{-s^2kt}.$$

Taking inverse transform, we get

$$\begin{aligned} u(x,t) &= F^{-1}\left[F(s)e^{-s^2kt}\right] \\ &= f(x)*F^{-1}(e^{-s^2kt}) \\ &= f(x)*\frac{e^{-\frac{x^2}{4kt}}}{\sqrt{2kt}} \\ &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\theta)e^{-\frac{(x-\theta)^2}{4kt}}\,d\theta. \end{aligned}$$

Putting $\frac{x-\theta}{2\sqrt{kt}} = \phi$, we get $\theta = x - 2\sqrt{kt}\,\phi$.

So,

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x-2\sqrt{kt}\,\phi)e^{-\phi^2}\,d\phi.$$

Conclusion

In this paper, Fourier transforms of some special functions, which can be used for solving one dimensional wave equation and heat equation, are found.

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